SUPPLEMENTARY MATERIAL

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S1. SCATTERING NETWORK MODEL

We consider the 2D oriented scattering network defined in the main text, and reproduced in Fig. 1 for the time period of two time-steps. We detail in this section the derivation of the evolution operator, its quasienergies and the center of mass trajectories showing Bloch oscillations.

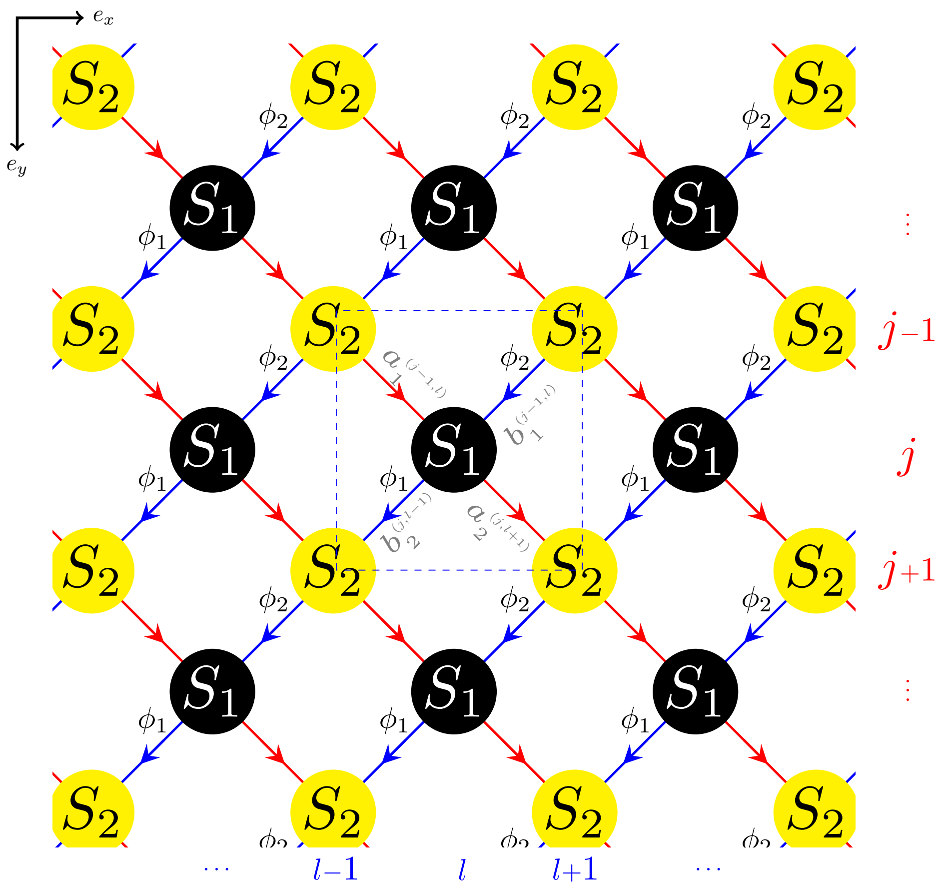

Figure 1. 2D oriented two-steps scattering network model with a preferential direction from top to bottom.

Derivation of the Floquet evolution operator

The oriented network shown in Fig. 1 is constituted of two distinct successive scattering nodes S_1 and S_2. The incoming arrow from left (right) toward the S_1 node is denoted by a_1 (b_1). It denotes a time evolution from the time step $j-1$ to time step j. Similarly, the outgoing arrows are denoted by a_2, b_2. These four oriented paths and the two scattering nodes constitute the unit cell of the network, which is emphasized with a dashed square in Fig. 1. The dynamics is then given by the relations

$$\begin{pmatrix} a_2(j,l+1) \\ b_2(j,l-1) \end{pmatrix} = S_1 \begin{pmatrix} a_1(j-1,l) \\ b_1(j-1,l) \end{pmatrix} \tag{1}$$

and

$$\begin{pmatrix} a_1(j-1,l) \\ b_1(j-1,l) \end{pmatrix} = S_2 \begin{pmatrix} a_2(j-2,l+1) \\ b_2(j-2,l-1) \end{pmatrix} , \tag{2}$$

which can be grouped together as

$$\begin{pmatrix} a_1(j-1,l) \\ b_1(j-1,l) \\ a_2(j,l+1) \\ b_2(j,l-1) \end{pmatrix} = \begin{pmatrix} 0 & S_2 \\ S_1 & 0 \end{pmatrix} \begin{pmatrix} a_1(j-1,l) \\ b_1(j-1,l) \\ a_2(j-2,l+1) \\ b_2(j-2,l-1) \end{pmatrix} . \tag{3}$$

The current scattering matrix notations correspond to that of the main text (Eq.(2)), as follows $a_1(j-1,l) = \alpha_l^{j-1}$, $b_1(j-1,l) = \beta_l^{j-1}$, similarly, $a_2(j,l+1) = \alpha_{l+1}^{j}$, $b_2(j,l-1) = \beta_{l-1}^{j}$.

Using translation symmetry of the scattering network, we can Fourier decompose as

$$\begin{pmatrix} a_m(j,l) \\ b_m(j,l) \end{pmatrix} = \sum_{k_x,k_y} \mathrm{e}^{i\vec{k}.(l\hat{e}_x + j\hat{e}_y)/2} \begin{pmatrix} a_m(k_x,k_y) \\ b_m(k_x,k_y) \end{pmatrix}, \quad m = 1,2 . \tag{4}$$

This gives,

$$\begin{pmatrix} a_1(k_x,k_y) \\ b_1(k_x,k_y) \\ a_2(k_x,k_y) \\ b_2(k_x,k_y) \end{pmatrix} = \begin{pmatrix} 0 & 0 & s_2^{11}\mathrm{e}^{ik_x/2}\mathrm{e}^{-ik_y/2} & s_2^{12}\mathrm{e}^{-ik_x/2}\mathrm{e}^{-ik_y/2} \\ 0 & 0 & s_2^{21}\mathrm{e}^{ik_x/2}\mathrm{e}^{-ik_y/2} & s_2^{22}\mathrm{e}^{-ik_x/2}\mathrm{e}^{-ik_y/2} \\ s_1^{11}\mathrm{e}^{ik_x/2}\mathrm{e}^{-ik_y/2} & s_1^{12}\mathrm{e}^{-ik_x/2}\mathrm{e}^{-ik_y/2} & 0 & 0 \\ s_1^{21}\mathrm{e}^{ik_x/2}\mathrm{e}^{-ik_y/2} & s_1^{22}\mathrm{e}^{-ik_x/2}\mathrm{e}^{-ik_y/2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1(k_x,k_y) \\ b_1(k_x,k_y) \\ a_2(k_x,k_y) \\ b_2(k_x,k_y) \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} \vec{a_1}(\vec{k}) \\ \vec{a_2}(\vec{k}) \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{S}_2(\vec{k}) \\ \mathcal{S}_1(\vec{k}) & 0 \end{pmatrix} \begin{pmatrix} \vec{a_1}(\vec{k}) \\ \vec{a_2}(\vec{k}) \end{pmatrix} \tag{6}$$

where $\vec{a_1}(\vec{k}) = \{a_1(\vec{k}), b_1(\vec{k})\}$ and $s_j^{m_1 m_2}$ $(m_1, m_2 = 1,2)$, are the scattering coefficients of the scattering matrix S_j. In the main text, we choose

$$S_j = \begin{pmatrix} \cos\theta_j & i\sin\theta_j \\ i\sin\theta_j & \cos\theta_j \end{pmatrix} . \tag{7}$$

although the calculations are independent of this specific form. Squaring Eq.(6) allows one to define the Floquet operators starting for different time origins as

$$\begin{pmatrix} \mathcal{S}_2(\vec{k})\mathcal{S}_1(\vec{k}) & 0 \\ 0 & \mathcal{S}_1(\vec{k})\mathcal{S}_2(\vec{k}) \end{pmatrix} = \begin{pmatrix} U_F^{21}(\vec{k}) & 0 \\ 0 & U_F^{12}(\vec{k}) \end{pmatrix} \tag{8}$$

Substituting Eq.(7) gives more specifically

$$U_F^{21}(\vec{k}) = \begin{pmatrix} \mathrm{e}^{-ik_y}(\mathrm{e}^{ik_x}\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) & i\mathrm{e}^{-ik_y}(\cos\theta_2\sin\theta_1 + \mathrm{e}^{-ik_x}\cos\theta_1\sin\theta_2) \\ i\mathrm{e}^{-ik_y}(\cos\theta_2\sin\theta_1 + \mathrm{e}^{ik_x}\cos\theta_1\sin\theta_2) & \mathrm{e}^{-ik_y}(\mathrm{e}^{-ik_x}\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \end{pmatrix} \tag{9}$$

$$U_F^{12}(\vec{k}) = \begin{pmatrix} \mathrm{e}^{-ik_y}(\mathrm{e}^{ik_x}\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) & i\mathrm{e}^{-ik_y}(\mathrm{e}^{-ik_x}\cos\theta_2\sin\theta_1 + \cos\theta_1\sin\theta_2) \\ i\mathrm{e}^{-ik_y}(\mathrm{e}^{ik_x}\cos\theta_2\sin\theta_1 + \cos\theta_1\sin\theta_2) & \mathrm{e}^{-ik_y}(\mathrm{e}^{-ik_x}\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \end{pmatrix} \tag{10}$$

Then, we add a phase ϕ to the b_j amplitudes, that is to the blue arrows in Fig. (1), such that $b_1 \to b_1\mathrm{e}^{i\phi_2}$ and $b_2 \to b_2\mathrm{e}^{i\phi_1}$. Then in Eq.(5), s_2^{12} and s_2^{22} will be multiplied by $\mathrm{e}^{i\phi_1}$, likewise, s_1^{12} and s_1^{22} are multiplied by $\mathrm{e}^{i\phi_2}$. That gives

$$U_F^{21}(\vec{k},\phi) = \mathrm{e}^{-ik_y} \begin{pmatrix} \mathrm{e}^{ik_x}\cos\theta_1\cos\theta_2 - \mathrm{e}^{i\phi_1}\sin\theta_1\sin\theta_2 & i(\mathrm{e}^{i\phi_2}\cos\theta_2\sin\theta_1 + \mathrm{e}^{-ik_x}\cos\theta_1\sin\theta_2) \\ i(\mathrm{e}^{i\phi_1}\cos\theta_2\sin\theta_1 + \mathrm{e}^{ik_x}\cos\theta_1\sin\theta_2) & \mathrm{e}^{-ik_x}\cos\theta_1\cos\theta_2 - \mathrm{e}^{i\phi_2}\sin\theta_1\sin\theta_2 \end{pmatrix} \tag{11}$$

$$U_F^{12}(\vec{k},\phi) = \mathrm{e}^{-ik_y} \begin{pmatrix} \mathrm{e}^{ik_x}\cos\theta_1\cos\theta_2 - \mathrm{e}^{i\phi_2}\sin\theta_1\sin\theta_2 & i(\mathrm{e}^{-ik_x}\cos\theta_2\sin\theta_1 + \mathrm{e}^{i\phi_1}\cos\theta_1\sin\theta_2) \\ i(\mathrm{e}^{ik_x}\cos\theta_2\sin\theta_1 + \mathrm{e}^{i\phi_2}\cos\theta_1\sin\theta_2) & \mathrm{e}^{-ik_x}\cos\theta_1\cos\theta_2 - \mathrm{e}^{i\phi_1}\sin\theta_1\sin\theta_2 \end{pmatrix} . \tag{12}$$

These two evolution operators describe the same physical system, and either of them can be used to compute the quasienergy spectrum and the winding numbers. The common phase factor $\exp\{-ik_y\}$ in Eqs.(11)-(12) is reminiscent of the preferential orientation of the network from top to bottom. This is the only k_y dependence of the evolution operator on the network. In the main text, the Floquet operator refers to U_F^{21} where this phase factor is factorized out, that is

$$U_F(k_x, \phi) \equiv U_F^{21}(k, \phi)\mathrm{e}^{ik_y} \tag{13}$$

and we set $k_x = k$ through out the paper. The eigenvalues of U_F are defined as $\mathrm{e}^{-i\epsilon T} \equiv \mathrm{e}^{i\varepsilon}$, where the dimensionless quasienergy ε is the quantity considered in the main text. Then the Floquet operator can usefully be factorized as

$$U_F = B_0(k)S_2D(\phi_2)B_1(k)S_1D(\phi_1), \tag{14}$$

where

$$B_1(k) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-ik} \end{pmatrix}, \; B_0 = \begin{pmatrix} \mathrm{e}^{ik} & 0 \\ 0 & 1 \end{pmatrix}, \; D_j = D(\phi_j) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{i\phi_j} \end{pmatrix}. \tag{15}$$

Alternative Hamiltonian formalism

We propose here an equivalent Hamiltonian formalism to the scattering model discussed in the main text and the previous section. An oriented network consisting of two scattering processes with two free propagations per period can be mapped onto a four time-step periodically driven tight-binding Hamiltonian. The scattering parameters of the network are related to the nearest-neighbors hopping terms J_i of a driven lattice, while the phase shifts accumulated during the free propagations are related to on-site potential J_i. The corresponding periodically driven 1D lattice over a time period T is depicted in Fig. 2 in the case of two-steps, as detailed in the main text. It is composed of two atoms per unitcell, denoted by A and B (similar to the two arrows, red and blue, entering each scattering node in Fig. 1 of main text), and is driven over 4 steps so that the stepwise Bloch Hamiltonian reads

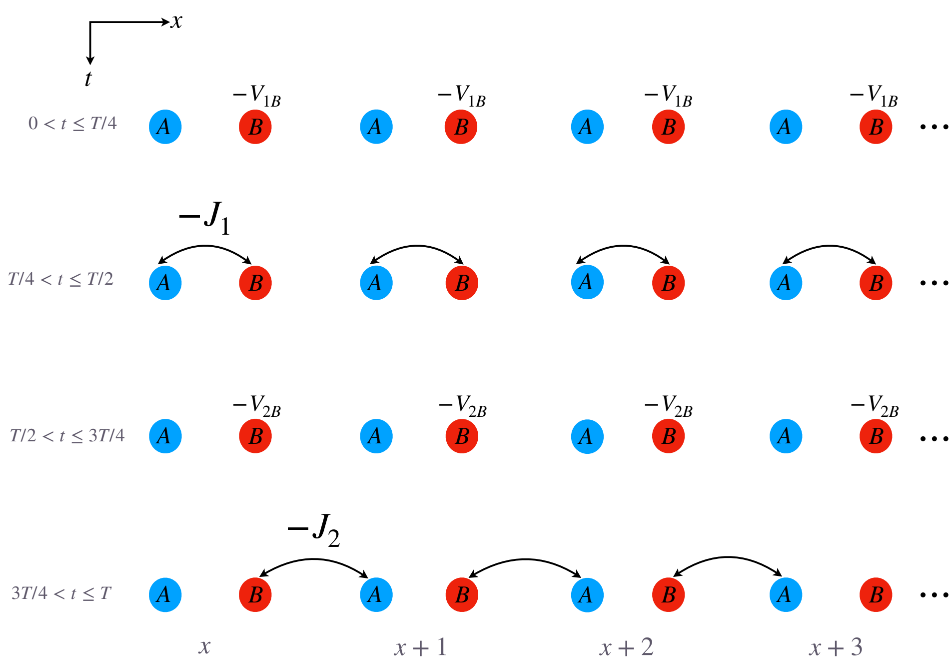

Figure 2. Periodically driven lattice model whose Floquet operator corresponds to that of the scattering network of the main text. The driving period is made of four steps during which either an onsite potential is switched on onto one sublattice (steps 1 and 3), or a hopping term is switched to dimerized the lattice (steps 2 and 4). The lattice spacing a is set to 1.

$$H(t,k_x) = \begin{cases} H_1(k_x) = \begin{pmatrix} 0 & 0 \\ 0 & -V_1 \end{pmatrix}, & 0 < t \leqslant t_1 \\ H_2(k_x) = \begin{pmatrix} 0 & -J_1 e^{-ik_x/2} \\ -J_1 e^{ik_x/2} & 0 \end{pmatrix} & t_1 < t \leqslant t_2 \\ H_3(k_x) = \begin{pmatrix} 0 & 0 \\ 0 & -V_2 \end{pmatrix} & t_2 < t \leqslant t_3 \\ H_4(k_x) = \begin{pmatrix} 0 & -J_2 e^{ik_x/2} \\ -J_2 e^{-ik_x/2} & 0 \end{pmatrix} & t_3 < t \leqslant T \end{cases} \tag{16}$$

Accordingly, a stepwise evolution operator U_j during the duration $\tau_j = t_j - t_{j-1}$ can be defined as

$$U_j(k_x) \equiv e^{-iH_j(k_x)\tau_j/\hbar}, \tag{17}$$

so that the evolution (Floquet) operator after one period is defined as $U_F(k_x) = U_4 U_3 U_2 U_1$. By setting

$$\phi_1 \equiv V_1\tau_1/\hbar \qquad \theta_1 \equiv J_1\tau_2/\hbar \quad \phi_2 \equiv V_2\tau_3/\hbar \quad \theta_2 \equiv J_2\tau_4/\hbar \tag{18}$$

the stepwise evolution operators read

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi_1} \end{pmatrix} \qquad U_2 = \begin{pmatrix} \cos\theta_1 & ie^{-ik_x/2}\sin\theta_1 \\ ie^{ik_x/2}\sin\theta_1 & \cos\theta_1 \end{pmatrix} \tag{19}$$

$$U_3 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \qquad U_4 = \begin{pmatrix} \cos\theta_2 & ie^{-ik_x/2}\sin\theta_2 \\ ie^{ik_x/2}\sin\theta_2 & \cos\theta_2 \end{pmatrix} \tag{20}$$

leading to the expression of the Floquet operator, as given in the main text (Eq.(3)).

Generalized inversion symmetry breaking

Considering the two parameters ϕ and k on the same footing allows us to define the generalized inversion symmetry $U_F(k,\phi) = \sigma_x U_F(-k,-\phi)\sigma_x$, where σ_x is the standard Pauli matrix. The existence of a net phase in the unit cell, (i.e. $\phi_1 + \phi_2 \neq 0$, with ϕ_1 and ϕ_2 proportional to ϕ) breaks this symmetry. This can be shown by symmetrizing the B_j matrices in Eq. (14) as

$$U_F(k,\phi) = B_0(k)\, S_2\, D(\phi_2)\, B_1(k)\, S_1\, D(\phi_1) = B(k)\, S_2\, D(\phi_2)\, B(k)\, S_1\, D(\phi_1) \tag{21}$$

where,

$$B(k) = \begin{pmatrix} e^{ik/2} & 0 \\ 0 & e^{-ik/2} \end{pmatrix} \tag{22}$$

and doing so as well for the D_j matrices, thus factorizing the net phase

$$U_F(k,\phi) = e^{i(\phi_1+\phi_2)/2} B(k)\, S_2\, \tilde{D}(\phi_2)\, B(k)\, S_1\, \tilde{D}(\phi_1) \tag{23}$$

where (as already defined as $U_{1,3}$ in Eq.(19) from Hamiltonian picture),

$$\tilde{D}(\phi_j) = \begin{pmatrix} e^{-i\phi_j/2} & 0 \\ 0 & e^{i\phi_j/2} \end{pmatrix} \tag{24}$$

Next we notice that $\sigma_x B(k)\sigma_x = B(-k)$ and $\sigma_x \tilde{D}(\phi_j)\sigma_x = \tilde{D}(-\phi_j)$ where we recall that ϕ_j is proportional to ϕ. Therefore, the net phase, in the phase factor in Eq. (23) prevents U_F to be inversion symmetric that is

$$\sigma_x U_F(k,\phi)\sigma_x \neq U_F(-k,-\phi)\ . \tag{25}$$

However, in the absence of a net phase, that is when $\phi_1 = -\phi_2$, this phase factor simplifies to 1 and one gets $\tilde{D}(-\phi_2) = e^{-i\phi_2/2} D(-\phi_2)$ and $\tilde{D}(-\phi_1) = e^{i\phi_2/2} D(-\phi_1)$, thus

$$\sigma_x U_F^{\phi_{net}=0}(k,\phi)\sigma_x = B(-k)\, S_2\, \tilde{D}(-\phi_2)\, B(-k)\, S_1\, \tilde{D}(-\phi_1) \tag{26}$$

$$= B(-k)\, S_2\, D(-\phi_2)\, B(-k)\, S_1\, D(-\phi_1) \tag{27}$$

$$= U_F^{\phi_{net}=0}(-k,-\phi) \tag{28}$$

so that the inversion symmetry is restored.

Derivation of the quasienergy bands

Let us derive here the quasienergy bands of the scattering network model with two time-steps as sketched in Fig. 1. This can be carried out analytically either by a direct diagonalization of U_F or equivalently by decomposing the evolution as in Ref [1]. Let us detail the second strategy. Using the same terminology as in the main text, where right going arrows are denoted with α and left going with β, then the time evolution is described by

$$\begin{aligned}\alpha_l^{j+1} &= (\cos\theta_1 \alpha_{l+1}^j + i\sin\theta_1 \beta_{l+1}^j)\mathrm{e}^{i\phi_1},\\ \beta_l^{j+1} &= \cos\theta_1 \beta_{l-1}^j + i\sin\theta_1 \alpha_{l-1}^j\end{aligned} \tag{29}$$

for the first step and

$$\begin{aligned}\alpha_{l-1}^{j+2} &= \left(\cos\theta_2 \alpha_l^{j+1} + i\sin\theta_2 \beta_l^j\right)\mathrm{e}^{i\phi_2},\\ \beta_{l-1}^{j+2} &= \cos\theta_2 \beta_{l-2}^j + i\sin\theta_2 \alpha_{l-2}^j\end{aligned} \tag{30}$$

for the second (final) step. Using Floquet-Bloch ansatz,

$$\begin{pmatrix}\alpha_l^j\\ \beta_l^j\end{pmatrix} = \begin{pmatrix}A\\ B\end{pmatrix}\mathrm{e}^{i\varepsilon j/2}\mathrm{e}^{ikl/2} \tag{31}$$

and substituting Eq.(29) in (30) using Eq.(31) gives the determinant problem

$$\mathrm{e}^{2i\varepsilon} - \left[\cos\theta_1\cos\theta_2\left(\mathrm{e}^{ik}\mathrm{e}^{i(\phi_1+\phi_2)} + \mathrm{e}^{-ik}\right) - \sin\theta_1\sin\theta_2\left(\mathrm{e}^{i\phi_1} + \mathrm{e}^{i\phi_2}\right)\right]\mathrm{e}^{i\varepsilon} + \mathrm{e}^{i(\phi_1+\phi_2)} = 0\ . \tag{32}$$

By rewriting the Eq.(32), we get the relation

$$\cos\left(\varepsilon - \frac{\phi_1+\phi_2}{2}\right) = \cos\theta_1\cos\theta_2\cos\left(k + \frac{\phi_1+\phi_2}{2}\right) - \sin\theta_1\sin\theta_2\cos\left(\frac{\phi_1-\phi_2}{2}\right),$$

that leads to

$$\varepsilon_\pm(k,\phi) = \pm\cos^{-1}\left[\cos\theta_1\cos\theta_2\cos\left(-k + \frac{\phi_1+\phi_2}{2}\right) - \sin\theta_1\sin\theta_2\cos\left(\frac{\phi_1-\phi_2}{2}\right)\right] + \left(\frac{\phi_1+\phi_2}{2}\right)\ . \tag{33}$$

We can finally substitute the general form for the ϕ's to $\phi_j = (m_j/n_j)\phi$, to get the expression

$$\varepsilon_\pm(k,\phi) = \pm\cos^{-1}\left[\cos\theta_1\cos\theta_2\cos\left(-k + \left[\frac{m_1}{n_1} + \frac{m_2}{n_2}\right]\frac{\phi}{2}\right) - \sin\theta_1\sin\theta_2\cos\left(\left[\frac{m_1}{n_1} - \frac{m_2}{n_2}\right]\frac{\phi}{2}\right)\right] + \left[\frac{m_1}{n_1} + \frac{m_2}{n_2}\right]\frac{\phi}{2} \tag{34}$$

for the quasienergy bands.

Derivation of the group velocities

Let us introduce the "synthetic group velocity" of the quasienergy band $\varepsilon_\pm$ as

$$v_\phi^\pm(k,\phi) \equiv \frac{\partial\varepsilon_\pm(k,\phi)}{\partial\phi} \tag{35}$$

Substituting the expression (34) of the quasienergy bands leads to

$$v_\phi^\pm(k,\phi) = \frac{1}{2}\Delta^+ \mp \frac{1}{2}\frac{\Delta^-\sin\theta_1\sin\theta_2\sin\left(\frac{1}{2}\phi\Delta^-\right) - \Delta^+\cos\theta_1\cos\theta_2\sin\left(k + \frac{1}{2}\phi\Delta^+\right)}{\sqrt{1 - \left(\cos\theta_1\cos\theta_2\cos\left(k+\frac{1}{2}\phi\Delta^+\right) - \sin\theta_1\sin\theta_2\cos\left(\frac{1}{2}\phi\Delta^-\right)\right)^2}} \tag{36}$$

where $\Delta^- \equiv \frac{m_1}{n_1} - \frac{m_2}{n_2}$ and $\Delta^+ \equiv \frac{m_1}{n_1} + \frac{m_2}{n_2}$. Similarly, the transverse group velocity is

$$v_k^\pm(k,\phi) \equiv \frac{\partial\varepsilon_\pm(k,\phi)}{\partial k} \tag{37}$$

$$v_k^\pm(k,\phi) = \pm\frac{\cos\theta_1\cos\theta_2\sin\left(k+\frac{1}{2}\phi\Delta^+\right)}{\sqrt{1 - \left(\cos\theta_1\cos\theta_2\cos\left(k+\frac{1}{2}\phi\Delta^+\right) - \sin\theta_1\sin\theta_2\cos\left(\frac{1}{2}\phi\Delta^-\right)\right)^2}}\ . \tag{38}$$

In the case $m_1 = n_1 = n_2 = 1, m_2 = -2$ considered in the main text, the quasienergy bands simplify to

$$\varepsilon_\pm(k,\phi) = \pm\cos^{-1}\left[\cos\theta_1\cos\theta_2\cos\left(k-\frac{\phi}{2}\right) - \sin\theta_1\sin\theta_2\cos\left(\frac{3\phi}{2}\right)\right] + \frac{\phi}{2} \tag{39}$$

which leads to the group velocities

$$v_k^\pm(k,\phi) = \pm\frac{\cos\theta_1\cos\theta_2\sin\left(k-\frac{\phi}{2}\right)}{\sqrt{1-\left(\cos\theta_1\cos\theta_2\cos\left(k-\frac{\phi}{2}\right)-\sin\theta_1\sin\theta_2\cos\left(\frac{3\phi}{2}\right)\right)^2}}, \tag{40}$$

$$v_\phi^\pm(k,\phi) = \frac{1}{2}\mp\frac{\frac{3}{2}\sin\theta_1\sin\theta_2\sin\left(\frac{3\phi}{2}\right)+\frac{1}{2}\cos\theta_1\cos\theta_2\sin\left(k-\frac{\phi}{2}\right)}{\sqrt{1-\left(\cos\theta_1\cos\theta_2\cos\left(k-\frac{\phi}{2}\right)-\sin\theta_1\sin\theta_2\cos\left(\frac{3\phi}{2}\right)\right)^2}}. \tag{41}$$

Note that when the quasienergy bands wind along the ϕ coordinate, then the quantity Δ^+ is non zero. Therefore, in that case, the numerator in the expression (38) of the transverse group velocity can change sign when varying ϕ, for any fixed value of k, thus giving rise to the wavepackets oscillations.

Along the same lines, we can calculate the motion of centre of mass from Eq.(40), for arbitrary k as,

$$\begin{aligned} X_c(t,k) &= \int_0^{\phi(t)} \mathrm{d}\phi\, v_k(\phi(\tau),k)\left(\frac{\partial\phi(\tau)}{\partial\tau}\right)^{-1}, \\ &= \gamma_0\int_0^t \mathrm{d}\tau\, v_k(\phi(\tau),k) \end{aligned} \tag{42}$$

where, in the last equation, we considered that ϕ varies linearly with time with a coefficient γ_0, (see Fig. 3(a) of the main text).

S2. WINDING NUMBER ν_ϕ FOR TWO TIME-STEPS EVOLUTIONS

Derivation of ν_ϕ.

Let us compute the winding number ν_ϕ defined in the main text, for two time-steps, where the Floquet operator $U_F(k,\phi)$ given in Eq. (14) with $\phi_1 = (m_1/n_1)\phi$ and $\phi_2 = (m_2/n_2)\phi$. When $|m_1/n_1| \neq |m_2/n_2|$, then

$$\nu_\phi = \frac{1}{2\pi i}\int_0^\Phi \mathrm{d}\phi\,\mathrm{Tr}\big[U_F^{-1}\partial_\phi U_F\big] \tag{43}$$

$$= \frac{1}{2\pi i}\int_0^\Phi \mathrm{d}\phi\,\mathrm{Tr}\Big[D_1^\dagger S_1^\dagger B_1^\dagger(k) D_2^\dagger S_2^\dagger B_0^\dagger(k)\partial_\phi\{D_1 S_1 B_1(k) D_2 S_2 B_0(k)\}\Big] \tag{44}$$

$$= \frac{1}{2\pi i}\int_0^\Phi \mathrm{d}\phi\,\mathrm{Tr}\Big[D_2^\dagger\partial_\phi D_2 + D_1^\dagger\partial_\phi D_1\Big] \tag{45}$$

where the period of Φ of the quasienergy in ϕ is inferred from the analytical expression (34). More precisely, it reads

$$\Phi = 2\pi\ \mathrm{LCM}\left[\frac{2}{\frac{m_1}{n_1}-\frac{m_2}{n_2}}, \frac{2}{\frac{m_1}{n_1}+\frac{m_2}{n_2}}\right] \tag{46}$$

where LCM stands for least common multiple. Replacing the D_j's matrices by their expression, one gets

$$\begin{aligned} \nu_\phi &= \frac{1}{2\pi i}\int_0^{2\pi\ \mathrm{LCM}\left[\frac{2}{\frac{m_1}{n_1}-\frac{m_2}{n_2}}, \frac{2}{\frac{m_1}{n_1}+\frac{m_2}{n_2}}\right]} \mathrm{d}\phi\,\mathrm{Tr}\left[i\frac{m_1}{n_1}+i\frac{m_2}{n_2}\right], \\ &= 2\ \mathrm{LCM}\left[\frac{1}{\frac{m_1}{n_1}-\frac{m_2}{n_2}}, \frac{1}{\frac{m_1}{n_1}+\frac{m_2}{n_2}}\right]\left(\frac{m_1}{n_1}+\frac{m_2}{n_2}\right), \\ &= 2\ \mathrm{LCM}\left[\frac{2n_1n_2}{m_1n_2-m_2n_1}, \frac{2n_1n_2}{m_1n_2+m_2n_1}\right]\left(\frac{m_1}{n_1}+\frac{m_2}{n_2}\right), \\ \nu_\phi &= \frac{\Phi}{2\pi}\left(\frac{m_1}{n_1}+\frac{m_2}{n_2}\right). \end{aligned} \tag{47}$$

Relation between ν_ϕ and the stationary points of the Bloch oscillations

Consider a situation where the quasienergy bands wind along ϕ, and let us count the number of stationary points $\frac{\mathrm{d}X_c}{\mathrm{d}t}$ in Eq.(42) over one Bloch period of oscillation. These points are determined by the vanishing of the group velocity v_k. Therefore, it suffices to find the number of roots in ϕ of Eq.(38), which are given by

$$\cos\theta_1 \cos\theta_2 \sin\left(\frac{1}{2}\phi\Delta^+\right) = 0 \tag{48}$$

This leads to

$$\begin{aligned} \frac{1}{2}\phi\Delta^+ &= p\pi, \qquad p \in \mathbb{Z} \\ \phi &= 2p\pi\frac{1}{\Delta^+}. \end{aligned} \tag{49}$$

which can be expressed in terms of the winding number given by Eq.(47) as

$$\phi = p\frac{\Phi}{\nu_\phi}. \tag{50}$$

Hence, over one oscillation period, p takes values from the set $\{1, ..., |\nu_\phi|\}$, and thus the group velocity vanishes ν_ϕ times. Then, following the same lines, one can easily check that the second derivative also vanishes at these same points. There are therefore ν_ϕ turning points per period of Bloch oscillation.

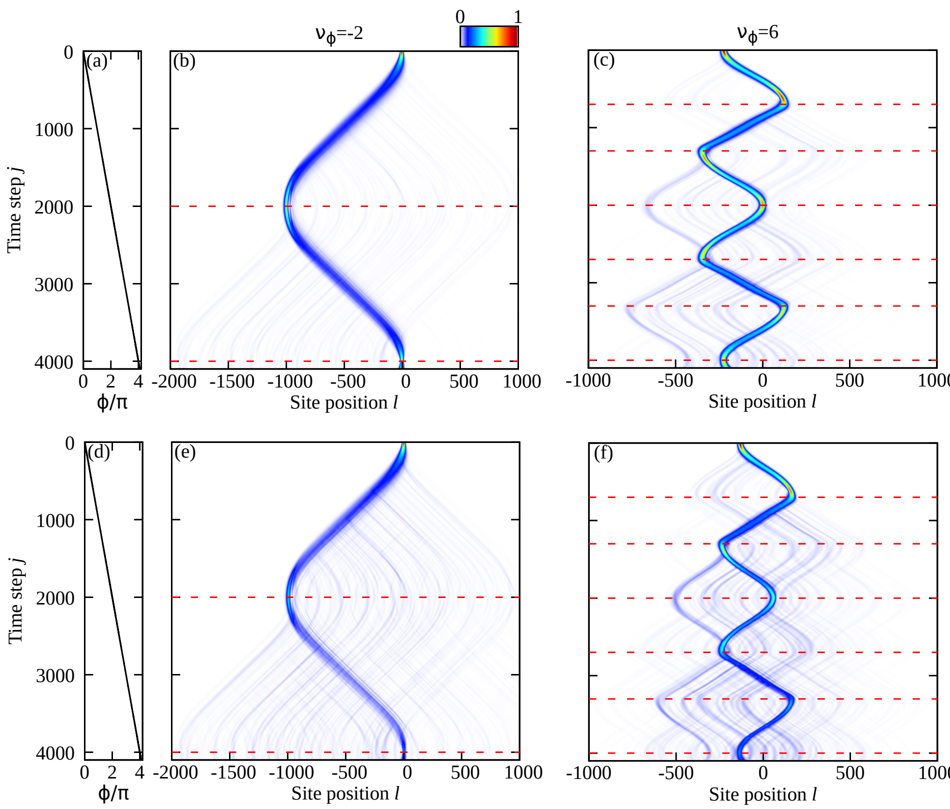

Figure 3. Intensity ($|\alpha_l^j|^2+|\beta_l^j|^2$) of a wavepacket, injected with a Gaussian shape when the values of θ_1 and θ_2 have a random disorder that is periodic with the Floquet period ($\theta_i + \delta_i$ with $i = 1, 2$ and δ_i a uniform distribution between $[-A, +A]$). Top: $A = 0.02$, bottom: $A = 0.04$. Left column: $\nu_\phi = -2$. Right column: $\nu_\phi = 6$. Other parameters have the same value than in the Fig. 3 of the main text. For visibility, horizontal red dashed lines are placed at each turning point.

Robustness of ν_ϕ under disorder

The Bloch oscillations emerged due to the underlying discrete translational symmetry of the lattice. However, these Bloch oscillations have topological origin where different values of ν_ϕ constitute the family. To check the robustness of these ν_ϕ, we introduce a disorder in the coupling parameters (θ_i with $i = 1, 2$); thus we break the discrete translational symmetry of the lattice. This is achieved by putting a disorder as $\theta_i + \delta_i$, where δ_i is a random number with a uniform distribution between the interval $[-A, +A]$. The dynamics of the wavepacket in these conditions is shown in figure 3, for the case of $\nu_\phi = -2$ and $\nu_\phi = 6$, and for a noise amplitude A of 0.02 and 0.04 (the other parameters are same as in Fig. 3 of the main text).
Besides the main oscillation, it appears, some "sub-branches" that seem to leave the main wavepacket. This effect is related to the presence of two bands in the system. The initial condition has been chosen to excite only one band of the system, however, with the variation of θ_i (induced by the noise term), the solution is not anymore restricted to two bands but becomes a multi-band problem, as a result, the other bands are also excited.

It induces the creation and evolution of "secondary" wavepackets. However, even with these wavepackets, the topological invariant ν_ϕ is preserved, since we can see that in all these wavepackets, the turning points appear at the same time (y-axis), and thus after one Bloch oscillation period, they have the same number of turning points, i.e. the same ν_ϕ.

S3. FICTITIOUS ELECTRIC FIELD IN THE NETWORK MODEL

Gauge transformation from a uniform electric field to winding bands with an adiabatic increase of ϕ

As pointed out by M. Wimmer et al. in Ref. [2] in the case of a single step model with an adiabatic increase of the phase factor $\phi(j) = \gamma_0\, j$, the gauge transformation:

$$\begin{aligned} \alpha_l^{j+1} &= \tilde{\alpha}_l^{j+1} \mathrm{e}^{-\frac{ilj\gamma_0}{2} + \frac{ij^2\gamma_0}{4} - \frac{ij\gamma_0}{4}} \\ \beta_l^{j+1} &= \tilde{\beta}_l^{j+1} \mathrm{e}^{-\frac{ilj\gamma_0}{2} + \frac{ij^2\gamma_0}{4} - \frac{ij\gamma_0}{4}} \end{aligned} \tag{51}$$

results in a set of equations in which the phase factor does not depend anymore on the time step, but presents a uniform gradient of phase :

$$\begin{aligned} \tilde{\alpha}_l^{j+1} &= (\cos\theta_j \tilde{\alpha}_{l+1}^j + i \sin\theta_j \tilde{\beta}_{l+1}^j) \mathrm{e}^{\frac{i\gamma_0 l}{2}} \\ \tilde{\beta}_l^{j+1} &= (i \sin\theta_j \tilde{\alpha}_{l-1}^j + \cos\theta_j \tilde{\beta}_{l-1}^j) \mathrm{e}^{\frac{i\gamma_0 l}{2}} \ . \end{aligned} \tag{52}$$

This set of equations corresponds to a scattering network subject to a homogeneous spatial phase gradient $V = E \cdot l$, where $E = \gamma_0/2$ can be interpreted as a homogeneous electric field along the l direction. When considering an initial wavepacket, the time evolution results in standard Bloch oscillations with period $T = 2\pi/E = 4\pi/\gamma_0$.

The same gauge transformation can be applied to each of the two-steps of the model with $n = 2$ discussed in the main text subject to an adiabatic increase of $\phi(j) = +\gamma_0 j$ (Fig. 3 of the main text).

Recall that in the first step $\phi_1(j) = (m_1/n_1)\gamma_0 j$ and in the second step $\phi_2(j) = (m_2/n_2)\gamma_0 j$. The transformation (51) results in a set of the Bloch-like Eqs.(52) for each of the two Floquet steps, each set characterized by a constant electric field in space. In the first step, the electric field is $E_1 = (m_1/n_1)\gamma_0/2$, and in the second step is $E_2 = (m_2/n_2)\gamma_0/2$. Therefore, we get back the Bloch oscillation picture in this case with an electric field that alternates between E_1 and E_2 at each subsequent step. The period T_B of the oscillations can be computed from the average electric field $(E_1 + E_2)/2$ over a full Floquet cycle.

The above discussion can also be simply understood from basic classical electrodynamics arguments[3, 4]. Indeed, in its most general form, an electric field can be expressed as $\mathbf{E} = -\nabla V + \partial \mathbf{A}/\partial t$. The Eq.(51) is the gauge transformation that transforms a gradient of spatial potential V, to a time-varying vector potential A.

Fictitious uniform electric field from a fictitious vector potential

The above discussion can also seen from the Floquet operator Eq.(14) using the simplification as in Eq.(23),

$$U_F(k,\phi) = \mathrm{e}^{i(\phi_1+\phi_2)/2}\, B(k)\, S_2\, \tilde{D}(\phi_2)\, B(k)\, S_1\, \tilde{D}(\phi_1) \ , \tag{53}$$

where B and $\tilde{D}$ are defined in Eq.(22) and Eq.(24). Then U_F can be further simplified by combining two diagonal matrices $\tilde{D}$ and B together, (also used in eq.(3) in the main text)

$$U_F(k,\phi) = \mathrm{e}^{i(\phi_1+\phi_2)/2}\, T_2\, S_2\, T_1\, S_1 \qquad (54)$$
$$T_j = \begin{pmatrix} \mathrm{e}^{i\tilde{k}_j} & 0 \\ 0 & \mathrm{e}^{-i\tilde{k}_j} \end{pmatrix}$$

where $\tilde{k}_j = k - \phi_j$. This form of the Floquet operator can be thought as describing a 1D lattice that is periodically driven in the presence of a time-varying vector potential $A(t) = Et$. The period T of this driving consists of two-steps where the vector potential redefines the Bloch momentum via Peierls' substitution. In the first step, for some fictitious charge q, $\phi_1 = qA_1 = qE_1 t$ that generates a fictitious electric field of magnitude E_1. Similarly, during the second step, $\phi_2 = qA_2 = qE_2 t$. This electric field translates in our case as $\phi_1 = -2\phi = -2qE$ and $\phi_2 = +\phi = +qE$. Thus, it gives rise to a net electric field $E_1 + E_2 \neq 0$, which is responsible for the Bloch oscillations.

S4. EXTENDED NETWORK MODEL FOR QUASIENERGY WINDING IN k

Time-step evolution equations for the scattering model with quasienergy winding in k

In the final part of the main text we introduce a model with long range hoppings that results in the winding of the quasienergy bands in the k direction. The model is sketched in Fig. 4(a) of the main text. The corresponding time step evolution equations are given by :

$$\begin{aligned} \alpha_l^{j+1} &= (\cos\theta_j \alpha_{l+l_2}^j + i\sin\theta_j \beta_{l+l_2}^j)\mathrm{e}^{i\phi_j} \\ \beta_l^{j+1} &= (i\sin\theta_j \alpha_{l+l_0}^j + \cos\theta_j \beta_{l+l_0}^j)\,, \end{aligned} \qquad (55)$$

$$\begin{aligned} \alpha_{l+l_3}^{j+2} &= (\cos\theta_{j+1} \alpha_l^{j+1} + i\sin\theta_{j+1} \beta_l^{j+1})\mathrm{e}^{i\phi_{j+1}} \\ \beta_{l+l_3}^{j+2} &= (i\sin\theta_{j+1} \alpha_{l+l_4}^{j+1} + \cos\theta_{j+1} \beta_{l+l_4}^{j+1})\,. \end{aligned} \qquad (56)$$

Here l_j is the link connecting the scattering nodes at time step $j+p-1$ to $j+p$, for some integer p. These l_j's can be defined in terms of r_j/s_j as

$$\frac{r_1}{s_1} = \frac{l_2 - l_3}{2}, \qquad \frac{r_2}{s_2} = \frac{l_2 - l_3 + l_0}{2}\ , \qquad (57)$$

If $\phi_1 = -\phi_2$, then there is no winding in ϕ, and the bands only wind in k. By combining both windings in k and ϕ, the drift and pseudo-oscillations of a wavepacket can be engineered together. Trajectories for the center of mass are shown in Fig. 4 for a fixed value of $\nu_\phi = -6$, and different values of ν_k, where thick curves are evaluated for a wavepacket centered at $k = 0$, and dashed curves for $k = 1$. The value of ν_ϕ can still be inferred from the number of turning points, as shown for $\nu_k = 8$.

Derivation of the winding number ν_k

For an arbitrary winding number in k and ϕ, the Floquet operator reads

$$U_F(k,\phi) = B_0(k_2)\, S_2\, D(\phi_2)\, B_1(k_1)\, S_1\, D(\phi_1) \qquad (58)$$

where, $k_j \equiv \left(\dfrac{r_j}{s_j}\right) k$ and $\phi_j \equiv \left(\dfrac{m_j}{n_j}\right)\phi$. This gives the quasienergies $\varepsilon_\pm(k,\phi)$

$$\varepsilon_\pm(k,\phi) = \pm\cos^{-1}\left[\cos\theta_1\cos\theta_2\cos\left(\delta^-\frac{k}{2} + \Delta^+\frac{\phi}{2}\right) - \sin\theta_1\sin\theta_2\cos\left(\delta^+\frac{k}{2} - \Delta^-\frac{\phi}{2}\right)\right] + \delta^+\frac{k}{2} + \Delta^+\frac{\phi}{2}\,, \qquad (59)$$

where $\delta^\pm \equiv \frac{r_1}{s_1} \pm \frac{r_2}{s_2}$. In the absence of a quasienergy winding along ϕ (or k), the corresponding Δ^+ (δ^+) terms vanish. The group velocity can then be derived exactly as

$$v_{g\pm}(\phi,k) = \frac{1}{2}\delta^+ \pm \frac{1}{2}\frac{\delta^+\sin\theta_1\sin\theta_2\sin\left(\delta^+\frac{k}{2} - \Delta^-\frac{\phi}{2}\right) - \delta^-\cos\theta_1\cos\theta_2\sin\left(\delta^-\frac{k}{2} + \Delta^+\frac{\phi}{2}\right)}{\sqrt{1 - \left(\cos\theta_1\cos\theta_2\cos\left(\delta^-\frac{k}{2} + \Delta^+\frac{\phi}{2}\right) - \sin\theta_1\sin\theta_2\cos\left(\delta^+\frac{k}{2} - \Delta^-\frac{\phi}{2}\right)\right)^2}}. \qquad (60)$$

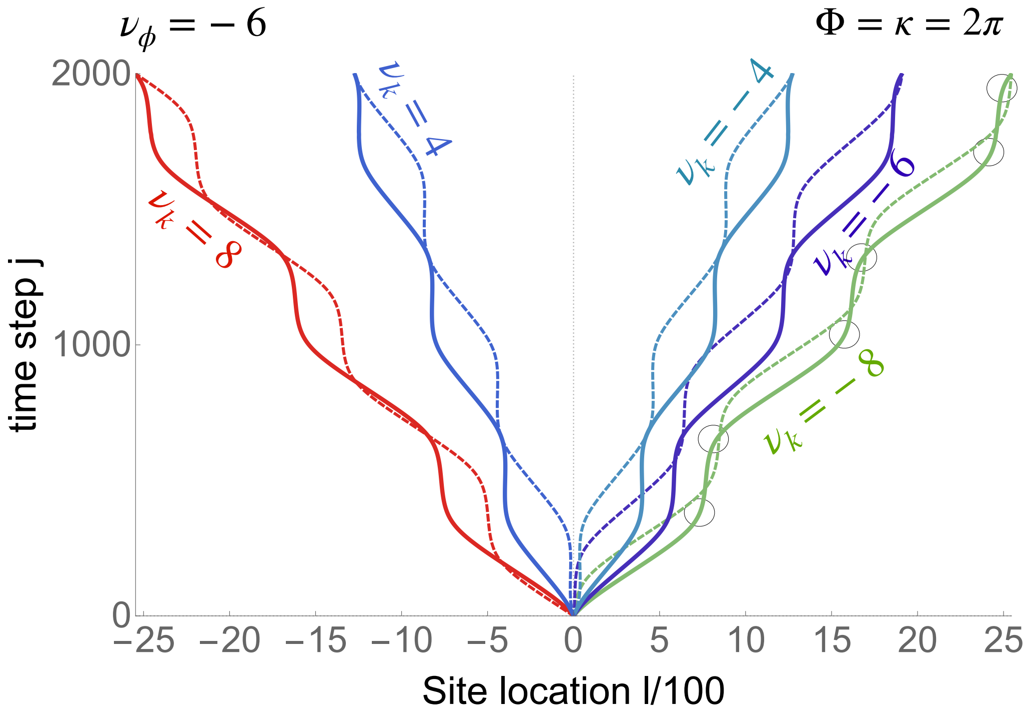

Figure 4. Trajectories in a two-steps model for bands with $\nu_\phi = -6$ for $\theta_1 = \pi/4$, $\theta_2 = \pi$, and different values of ν_k, ranging from $\nu_k = +8$ to $\nu_k = -8$, where the thick curves are for $k = 0$, and the dashed curves are for $k = 1$. Thus, changing k merely shifts the curve in vertical direction. Moreover, $|\nu_\phi|= 6$ can still be read even in this case of winding in both k and ϕ from the number of turning points, irrespective of the initial value of k, as shown with little circles for $\nu_k = -8$ (in green). Here, ϕ increases as $\phi(j) = \gamma_0 j$, with $\gamma_0 = 2\pi/2000$

Likewise, the winding number in k can be computed similar to ν_ϕ as

$$\begin{aligned}
\nu_k &\equiv \frac{1}{2\pi i}\int_0^\kappa \mathrm{d}k\,\mathrm{Tr}\big[U_F^{-1}\partial_k U_F\big] && (61)\\
&= \frac{1}{2\pi i}\int_0^\kappa \mathrm{d}k\,\mathrm{Tr}\left[i\frac{r_1}{s_1} + i\frac{r_2}{s_2}\right],\\
&= 2\ \mathrm{LCM}\left[\frac{1}{\frac{r_1}{s_1}-\frac{r_2}{s_2}}, \frac{1}{\frac{r_1}{s_1}+\frac{r_2}{s_2}}\right]\left(\frac{r_1}{s_1}+\frac{r_2}{s_2}\right),\\
&= 2\ \mathrm{LCM}\left[\frac{2s_1s_2}{r_1s_2-r_2s_1}, \frac{2s_1s_2}{r_1s_2+r_2s_1}\right]\left(\frac{r_1}{s_1}+\frac{r_2}{s_2}\right),\\
\nu_k &= \frac{\kappa}{2\pi}\left(\frac{r_1}{s_1}+\frac{r_2}{s_2}\right) && (62)
\end{aligned}$$

where $\kappa = 2\pi\ \mathrm{LCM}\left[\frac{2}{\frac{r_1}{s_1}-\frac{r_2}{s_2}}, \frac{2}{\frac{r_1}{s_1}+\frac{r_2}{s_2}}\right]$.

S5. RELATION BETWEEN THE WINDING NUMBER ν_k AND THE QUANTIZED DRIFT Δx

Let us introduce the mean current over a Floquet period T as

$$J \equiv \int_0^T \frac{\mathrm{d}t}{T} j(t) \tag{63}$$

that we express in terms of the instantaneous current $j(t)$

$$j(t) = \int_0^\kappa \frac{\mathrm{d}k}{\kappa} \langle\psi(k,t)| \frac{\mathrm{d}x(t)}{\mathrm{d}t} |\psi(k,t)\rangle, \tag{64}$$

where $|\psi(k,t)\rangle$ is an arbitrary evolving Bloch state, i.e. $|\psi(k,t)\rangle = U(k;t,0)|\psi(k,0)\rangle$ with $U(k;t,0)$ the Block evolution operator from time $t=0$ to arbitrary time $t<T$. Rewriting $U(k;t,0) = U(k;t,T)U(k;T,t)$, and using the relation $i\frac{\partial U_F}{\partial k} = \int_0^T \mathrm{d}t\, U(k;T,t)\, \frac{\partial H(k,t)}{\partial k}\, U(k;t,0)$, where $H(k,t+T) = H(k,t)$ is the periodically driven Bloch Hamiltonian, the mean current can be written in terms of the Floquet operator only

$$J = -\frac{2\pi/\kappa}{T} \int_0^\kappa \frac{\mathrm{d}k}{2\pi i} \langle\psi(k,0)|\, U_F^{-1} \frac{\partial U_F}{\partial k}\, |\psi(k,0)\rangle \tag{65}$$

Equivalently, one assigns a mean displacement $\Delta x = TJ$ to this current.

Adiabatic regime

Consider an instantaneous eigenstate $\varphi^{(n)}(k,t)$ of $H(k,t)$, such that $\psi(k,0) = \varphi^{(n)}(k,t=0)$. In the adiabatic limit, $\varphi^{(n)}(k,t)$ remains an eigenstate of $H(k,t)$ at each time. After a cycle $t: 0 \to T$, $\varphi^{(n)}(k,0)$ can only acquire a phase, which is by definition the quasienergy $\epsilon_n T$. It is thus an eigenstate of the Floquet operator, which therefore allows the spectral decomposition $U_F = \sum_n^N \exp(-i\epsilon_n T)|\varphi^{(n)}(k,0)\rangle\langle\varphi^{(n)}(k,0)|$, so that the mean current (65) simply becomes

$$J_{ad}^{(n)} = -\frac{2\pi/\kappa}{T} \int_0^\kappa \frac{\mathrm{d}k}{2\pi} \frac{\partial \varepsilon_n}{\partial k} \tag{66}$$

where the dimensionless quasienergy $\varepsilon_n = -\epsilon_n T$ corresponds to that of the main text. The adiabatic pumped current is quantized in terms of the quasienergy winding numbers along the k direction, as found in Ref [5]. As pioneered by Thouless [6], this quantization can be consistently rephrased in terms of the Chern numbers C_n of the adiabatically driven eigenstates $\varphi^{(n)}(k,t)$ that defined a $U(1)-$fiber bundle over the two-dimensional torus span by (k,t), assuming the instantaneous energy band $E_n(k,t)$ (the eigenvalue of $H(k,t)$) remains well separated from the other bands. One way to see the connection between the two topological points of view consists in identifying the quasienergy in terms of the dynamical phase and the geometrical Berry phase

$$\epsilon_n T = E_n T + i \int_0^T \mathrm{d}t \langle\varphi^{(n)}(k,t)|\partial_t|\varphi^{(n)}(k,t)\rangle \ . \tag{67}$$

Taking, the "winding" of this expression, that is applying $\int \mathrm{d}k\partial_k$ yields

$$-\int \frac{\mathrm{d}k}{2\pi}\partial_k\varepsilon_n = i\int \frac{\mathrm{d}k}{2\pi}\int_0^T \partial_k \mathrm{d}t\langle\varphi^{(n)}(k,t)|\partial_t|\varphi^{(n)}(k,t)\rangle \tag{68}$$

since the instantaneous energy band $E_n(k,t)$ cannot wind along k. Inserting the relation $\partial_k\langle\varphi|\partial_t\varphi\rangle = \langle\partial_k\varphi|\partial_t\varphi\rangle - \langle\partial_t\varphi|\partial_k\varphi\rangle + \partial_t\langle\varphi|\partial_k\varphi\rangle$, into the right-hand side of (68), the quasienergy winding reads

$$-\int \frac{\mathrm{d}k}{2\pi}\partial_k\varepsilon_n = \frac{1}{2\pi}\int_0^T \int \mathrm{d}k F_{k,t}^{(n)} + \int_0^T \partial_t Z^{(n)}(t) \tag{69}$$

where $F_{k,t}^{(n)}$ is the Berry curvature and $Z^{(n)}(t)$ is the time-dependent Zak phase of the instantaneous state $\varphi^{(n)}$, i.e. $Z^{(n)}(t) \equiv i\int \mathrm{d}k\langle\varphi^{(n)}(k,t)|\partial_k\varphi^{(n)}(k,t)\rangle$. After an adiabatic cycle, one has $Z^{(n)}(T) = Z^{(n)}(0)$, which leads to the relation between the winding number $\nu_k^{(n)}$ of the quasienergy band n in the k direction and the Chern number C_n of the adiabatically periodically driven Bloch eigenstate $\varphi^{(n)}(k,t)$

$$-\int \frac{\mathrm{d}k}{2\pi}\partial_k\varepsilon_n = C_n \ . \tag{70}$$

When the α lowest bands are filled, the adiabatic pumped current reads

$$\bar{J}_\alpha = \sum_{n=1}^{\alpha} J_{ad}^{(n)} = \frac{2\pi/\kappa}{T}\sum_{n=1}^{\alpha} C_n \tag{71}$$

in agreement with the famous Thouless result on adiabatic pumping. Clearly, if all the bands are filled, then $\bar{J}_N = \nu_k = 0$ owing to the vanishing sum of the Chern numbers over all the bands.

Non-adiabatic regime

We now consider the case where the instantaneous eigenstates $\varphi^{(n)}(k,t)$ do not remain eigenstates during the evolution, so that the total mean current $\bar{J}_N$ reads

$$\bar{J}_N = \sum_{n=1}^{N} \int_0^T \frac{\mathrm{d}t}{T} \int_0^\kappa \frac{\mathrm{d}k}{\kappa} \langle \varphi^{(n)}(k,t)| \frac{\mathrm{d}x(t)}{\mathrm{d}t} |\varphi^{(n)}(k,t)\rangle \tag{72}$$

$$= -\frac{2\pi/\kappa}{T} \int_0^\kappa \frac{\mathrm{d}k}{2\pi i} \,\mathrm{tr}[U_F^{-1} \frac{\partial U_F}{\partial k}] \tag{73}$$

$$= -\frac{2\pi/\kappa}{T} \nu_k \tag{74}$$

with $\nu_k \in \mathbb{Z}$ is the winding number of the map $k \in S^1 \to U_F \in U(N)$, and whose another expression is given by Eq.(9) of the main text. Moreover, since $\mathrm{tr}\, U_F^{-1} \frac{\partial U_F}{\partial k} = \frac{\partial}{\partial k} \ln \det U_F$, this winding number reads

$$\nu_k = \sum_{n=1}^{N} \int_0^\kappa \frac{\mathrm{d}k}{2\pi} \frac{\partial(-\epsilon_n T)}{\partial k} = \sum_{n=1}^{N} \int_0^\kappa \frac{\mathrm{d}k}{2\pi} \frac{\partial \varepsilon_n}{\partial k} \tag{75}$$

so that the mean displacement $\Delta x = T \bar{J}_N$, after P periods T, can be expressed in terms of the sum of the winding of the quasienergies of all the bands

$$\Delta x = -P \frac{2\pi}{\kappa} \sum_{n=1}^{N} \int_0^\kappa \frac{\mathrm{d}k}{2\pi} \frac{\partial \varepsilon_n}{\partial k} \tag{76}$$

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